20070510

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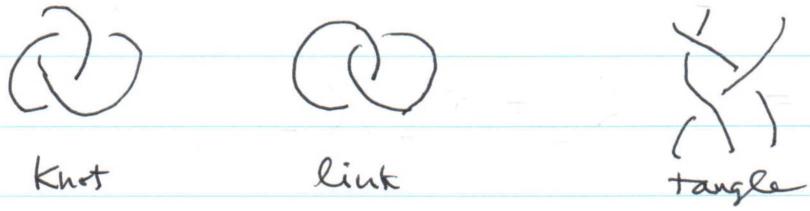
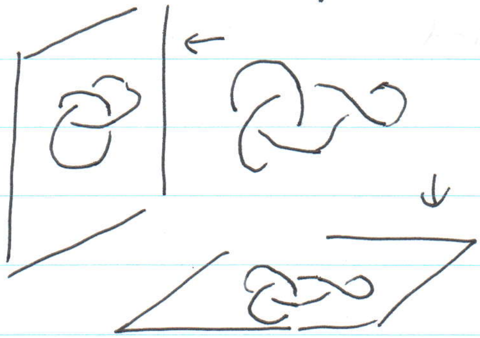
Lecture 1

Goal: Introduce the link invariant known as the Jones polynomial and explain its extension to tangles

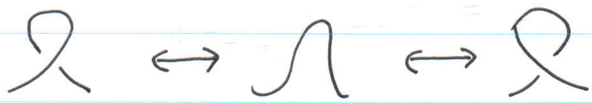


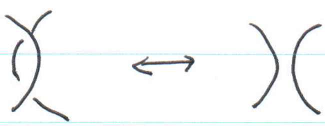
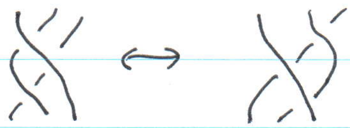
Assign data to planar projection such that it does not depend on the projection chosen



Th (Reidemeister)

Two link diagrams represent isotopic links iff one can be obtained from the other by a finite sequence of moves

(R1) 

(R2)  (R3) 

Kauffman bracket:

If L is a link diagram, let

$[L] \in \mathbb{Z}[a, a^{-1}]$ be defined by

(i) $[\bigcirc] = 1$

(ii) $[\bigcirc \sqcup L] = (-a^2 - a^{-2})[L]$ where $L \neq \phi$

(iii) $[\times] = a[\asymp] + a^{-1}[)(]$

This is almost a link invariant. It is invariant under R2 and R3, but not R1

If we demand that L is oriented

n_- $\quad\quad$ n_+

$\shortparallel$

number of negative crossings

Then for a link diagram L the polynomial

$$f[L] = \underbrace{(-a)^{-3(n_+ - n_-)}}_{\text{scaling}}[L]$$

Jones used $J(L) = f[L]\big|_{a = t^{-1/4}}$

(R1) $[\Omega] = a[\Omega] + a^{-1}[\Omega]$ by (iii)

$= a(-a^{2}-a^{-2})[\Omega] + a^{-1}[\Omega] = -a^{3}[\Omega]$

orient

$n_+ = 1,\ n_- = 0$

$f[\Omega] = -a^{-3}(-a^{3})f[\Omega] = f[\Omega]$

$[\Omega] = a[\Omega] + a^{-1}[\Omega]$

$= a[\Omega] + a^{-1}(-a^{2}-a^{-2})[\Omega] = -a^{-3}[\Omega]$

n_+

$f[\Omega] = (-a^{3})(-a^{-3})f[\Omega]$

(R2) $[\ \ ] = a[\ \ ] + a^{-1}[\ \ ]$

$n_+ = 1$
$n_- = 1$

$= a(-a^{-3})[\asymp] + a^{-1}\left(a[)(] + a^{-1}[\asymp]\right)$

$= [)(]$

<u>Exercise</u> Prove (R3)

Hint Use R2

<u>Warning</u> : Khovanov uses the "scaled" Kauffman bracket $\langle L\rangle$

(i) $\langle \emptyset \rangle = 1$

(ii) $\langle O \sqcup L \rangle = (q+q^{-1})\langle L\rangle$

(iii) $\langle \times \rangle = \langle \asymp \rangle - q\langle)(\rangle$

The two are related by $q \to -a^{2}$ $\hat{J}(L) = (-1)^{n_-} q^{n_+-2n_-}\langle L\rangle$

The Jones poly. in Khovanov's normalization

$$\hat{J}(L)_{(\xi=-a^2)} = f[L](-a^2-a^{-2}) \quad \lrcorner$$

<u>Cor.</u> A knot K whose Jones polynomial is not symmetric under $a \to a^{-1}$ is distinct from its mirror (chiral)

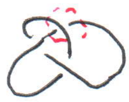
mirror of L  $\leftrightarrow$ 

Example

1) [] $= a$ [] $+ a^{-1}$ []

$= -a^4 - a^{-4}$

from our (RI) calculation

2) [] $= a$ [] $+ a^{-1}$ []

$= a(-a^4 - a^{-4}) + a^{-1}(a^{-3})^2$

Ex 1) from Ω calc.

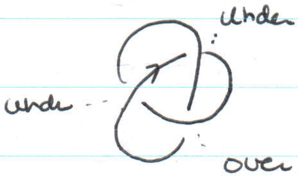
$= -a^5 - a^{-3} + a^{-7}$

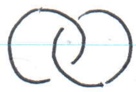
Not symmetric under $a \to a^{-1}$

$\Rightarrow$ trefoil is chiral.

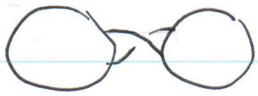
Application : Tait conjecture

A link diagram is alternating if the crossing change over under as you go around the link





Tait conj.

A reduced (not of the form )
alternating link
diagram is minimal (i.e. least number of crossings)

Note : there may be many diagrams with the same number of crossings

Proof · width of the Jones polynomial $\leqq 4$ times the number of crossings

difference of the highest and lowest power of a

· width of the Jones polynomial for a reduced alternating diagram $= 4$ times 〃 //

Different proof using Khovanov hom.

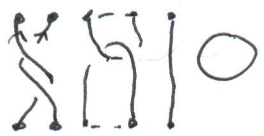
Relationship to $U_q(sl_2)$

We can extend the Jones polynomial to a functor

Tang : $\longrightarrow$ representation category of the Hopf alg $U_q(sl_2)$

objects $\overbrace{\bullet\ \bullet\ \cdots\ \bullet}^{n}$

morphisms



$\overbrace{\bullet\ \bullet\ \cdots\ \bullet}^{n} \overset{J}{\longmapsto} V^{\otimes n}$ $\qquad V$: 2-dim. irrep. of $U_q(sl_2)$

$\overset{J}{\longmapsto}$ $V^{\otimes 2}$

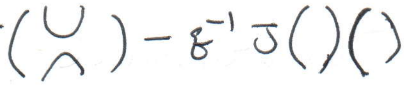
$\uparrow$ $\leftarrow$ intertwines the $U_q(sl_2)$ actions

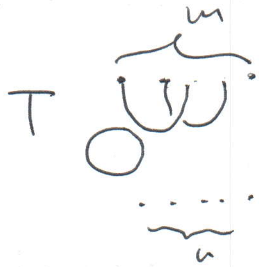
k

ground ring

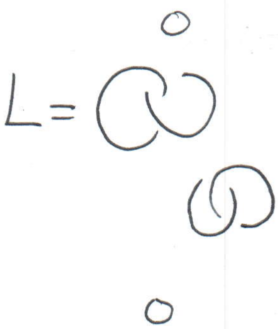
$\overset{J}{\longmapsto}$ $k \uparrow V^{\otimes 2}$

$\overset{J}{\longmapsto}$





$\longrightarrow$ $V^{\otimes m} \xleftarrow{J(T)} V^{\otimes n}$ ... operator which intertwines the action



$\longmapsto$ $k \xleftarrow{J(L)} k$ $\qquad k = \mathbb{Z}[q, q^{-1}]$